A PYTHAGOREAN FUNCTIONAL EQUATION.*

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1. **Introduction.** It is well known that

$$|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y. \tag{1}$$

This relation can also be written

$$|\sin(x+iy)|^2 = |\sin x|^2 + |\sin iy|^2, \tag{2}$$

or, in other words, the function $\sin z$ $(z = x+iy)$ satisfies the functional equation

$$|f(z)|^2 = |f(x)|^2 + |f(iy)|^2, \tag{3}$$

a *Pythagorean Theorem* in the theory of functions.

It is easy to see that $\sin z$ is not the only solution of (3). Evidently $C\sin z$ will also satisfy where C is an arbitrary complex constant. A special solution is found to be given by Cz. It is not hard to verify that

$$C\frac{\sin az}{a} \tag{4}$$

is a solution of (3) where C is an arbitrary constant and a is either real or purely imaginary. We shall prove that this is the general solution, if we restrict ourselves to analytic solutions.

In § 2 we reduce the problem to the solving of a differential equation in two independent and two dependent variables. This equation we integrate by a specialization of one of the independent variables which is equivalent to assuming the solutions of (3) single-valued and analytic in the neighbourhood of the origin and at that point.

In § 3 this assumption is justified in different ways. This section also contains an *a priori* discussion of the singularities of solutions of the functional equation.

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* Presented to the American Mathematical Society, October 28, 1922. The results of this paper may be considered trivial, but the author believes that the methods used are of some interest and may find applications to more general problems.

2. The reduction. Let us introduce

$$(5)\qquad \begin{gathered}[f(z)]^2 = F(z);\\ |F(z)| = R(z); \qquad |F(x)| = G(x); \qquad |F(iy)| = H(y),\end{gathered}$$

which carries (3) over into the form

$$(6)\qquad R(z) = G(x) + H(y),$$

which is easy to solve. We can define $G(x)$ and $H(y)$ arbitrarily and $R(z)$ will be defined by (6). But when is the solution so obtained, the absolute value of an analytic function?

It is necessary that log $[R(z)]$ *is a harmonic function.* Hence

$$(7)\qquad \Delta\,\{\log[R(z)]\} = 0,$$

where Δ stands for the Laplacian. If we substitute for $R(z)$ its expression (6), form the Laplacian and simplify, we get

$$(8)\qquad [G(x)+H(y)]\;[G''(x)+H''(y)] - [G'(x)]^2 - [H'(y)]^2 = 0,$$

where the primes denote differentiation with respect to the argument of the function in question. Here we have a mixed differential equation containing two unknown functions $G(x)$ and $H(y)$. In order to obtain $G(x)$ we specialize y. The most natural specialization would certainly be to equate y to 0. This is allowable, at least if the solutions of (3) are single-valued and analytic at the origin.

We assume that this is the case and shall justify our assumption in the next section. Suppose a solution $f(z)$ of (3) is given by a power series

$$(9)\qquad f(z) = c_0 + c_1 z + c_2 z^2 + \cdots,$$

convergent for small values of z; $f(z)$ being analytic we can put $z = 0$ in equation (3) obtaining

$$(10)\qquad f(0) = c_0 = 0.$$

Further put $|c_1| = c$, then

$$(11)\qquad R(z) = c^2|z|^2\{1 + p(|z|^2)\},$$

where $p(|z|^2)$ is a power series in $|z|^2$ vanishing with z. Hence

$$H(0) = H'(0) = 0; \qquad H''(0) = 2c^2. \tag{12}$$

Substituting $y = 0$ in (8) we obtain

$$G(x)\, G''(x) - [G'(x)]^2 + 2c^2\, G(x) = 0. \tag{13}$$

If we put

$$G(x) = u^2, \tag{14}$$

the differential equation (13) is carried over into

$$uu'' - (u')^2 + c^2 = 0. \tag{15}$$

This equation is integrable by elementary methods and the general solution is found to be

$$u = c\, \frac{\sin a(x-\alpha)}{a}, \tag{16}$$

where a and α are arbitrary constants. The limiting value for $a \to 0$, namely $c(x-\alpha)$, is also a solution. We want to get the solution which vanishes at the origin. Hence $\alpha = 0$. Further $u^2 = G(x) > 0$ which shows that u must be a real function of x. c being real, we have to take a either real or purely imaginary. Thus

$$G(x) = |f(x)|^2 = c^2 \frac{\sin^2 ax}{a^2}. \tag{17}$$

In a similar manner we show that

$$H(y) = |f(iy)|^2 = d^2 \frac{\sin^2 by}{b^2}. \tag{18}$$

But $f(z)$ is analytic at the origin, thus

$$c = d, \qquad b = ia. \tag{19}$$

Hence

$$|f(z)|^2 = c^2\left[\frac{\sin^2 ax}{a^2} + \frac{\sinh^2 ay}{a^2}\right] = c^2 \left|\frac{\sin az}{a}\right|^2. \tag{20}$$

If the absolute value of an analytic function is known in a simply connected region, then the argument is determined uniquely up to an additive real constant. Consequently

$$f(z) = C\frac{\sin az}{a}, \tag{21}$$

where C is an arbitrary complex constant, $|C| = c$, and a is a constant, either real or purely imaginary. When $a = 0$, formula (21) is understood to mean $f(z) = Cz$.

3. **The singularities.** We have assumed in § 2 that an arbitrary solution, $f(z)$, of (3) is single-valued and analytic at the origin. This assumption can be justified in different ways.

We can use equation (8) for this purpose. Suppose that every analytic solution of (3) is single-valued and analytic in the neighbourhood of some point on the axis of imaginaries. This point may, of course, differ from one solution to another. Suppose that for the solution $f(z)$ we may take $y = y_0$ and assume

$$H(y_0) = A, \quad H'(y_0) = B, \quad H''(y_0) = C.$$

Here $A > 0$, B and C are real. The equation (8) becomes

$$[G(x)+A][G''(x)+C]-[G'(x)]^2-B^2 = 0. \tag{22}$$

This equation is also integrable in terms of elementary functions. The general integral is of the form

$$G(x) = a\sin(bx+c)+d, \tag{23}$$

where two of the constants are expressible in terms of the other two and A, B, C. Their values are of no interest in this connection.

Supposing $f(z)$ single-valued and analytic even in the neighbourhood of $z = x_0$, we obtain in a similar manner

$$H(y) = \alpha\sin(\beta y+\gamma)+\delta \tag{24}$$

where, of course, α, β, γ, δ are not independent of a, b, c, d. Substitute the expressions (23) and (24) in equation (8) and we obtain

$$\begin{aligned} -a^2b^2-\alpha^2\beta^2-(d+\delta)\,[a\,b^2\sin(bx+c)+\alpha\,\beta^2\sin(\beta y+\gamma)] \\ -a\,\alpha\,(b^2+\beta^2)\sin(bx+c)\sin(\beta y+\gamma) = 0. \end{aligned} \tag{25}$$

From this relation we get

1° $a^2 b^2 + \alpha^2 \beta^2 = 0$. (Put $bx + c = 0$, $\beta y + \gamma = 0$).

2° $d + \delta = 0$. (Put $bx + c = 0$).

3° $b^2 + \beta^2 = 0$. (Put x and y arbitrary).

Thus

$$R(z) = a[\sin(bx + c) + \sin(biy + \gamma)]. \tag{26}$$

If we remember that $R(z)$ is positive for $x = 0$, y close to y_0, and for $y = 0$, x close to x_0, we conclude that ab and ac are real, and, moreover, a, b, c either all real or all purely imaginary. If we take b real, we find further $c = (4m + 3)\pi/2$, $\gamma = (4n + 1)\pi/2$. Then (26) takes the form

$$\begin{aligned} R(z) &= +|a|[-\cos bx + \cosh by] \\ &= 2|a|\left[\sin^2\left(\frac{b}{2}x\right) + \sinh^2\left(\frac{b}{2}y\right)\right], \end{aligned} \tag{27}$$

or the same result as in formula (20).

Finally, it is possible to gain some information concerning the possible singular points of analytic solutions of (3), supposed single-valued within their domain of existance, without solving the equation. The following principle will help us to discuss the singular points.

A solution of (3) *such that every simply connected region in the plane, no matter how small, contains an interior region, also simply connected, in which the solution is single-valued and analytic, can not have a finite singular point* $z_0 = x_0 + iy_0$, *such that it is possible to find a monotonic point-set* $x_1, x_2 \ldots x_n \ldots$ *or* $y_1, y_2, \ldots, y_n \ldots$ *with* $\lim_{n\to\infty} x_n = x_0$ *or* $\lim_{n\to\infty} y_n = y_0$ *and* $\lim_{n\to\infty} f(x_n) = \infty$ *or* $\lim_{n\to\infty} f(y_n) = \infty$. *In special no limit is possible at the origin other than* 0.

As the function $f(z)$ exists in the whole plane the line $x = x_0$ can not be a singular line throughout. Hence we can find a value $z = x_0 + iY_0$ in the neighbourhood of which $f(z)$ is analytic. But

$$\lim_{n\to\infty} |f(x_n + iY_0)|^2 = \lim_{n\to\infty} |f(x_n)|^2 + |f(iY_0)|^2,$$

which leads to a contradiction, no matter what value $f(iY_0)$ has. The special statement for the origin is proved in the same manner.

This principle excludes poles for the solutions. It also excludes isolated essentially singular points, because we can always find a Jordan curve along which $f(z)$ tends to the limit ∞.* We project the curve on the axes. $f(z)$ can not remain bounded on these projections, hence we can pick out a point set as required. The extension to non-isolated points is possible as long as $f(z)$ can not remain bounded on its singular points. Hence the principle is applicable if the set of singular points is enumerable but generally not if the set is perfect.†

* Proof by Valiron: Démonstration de l'existance pour les fonctions entières, de chemins de détermination infinie, Paris, C. R., 166 (1918), pp. 382–384, for entire functions. The extension to general essentially singular points is obvious. Also W. Gross: Über die Singularitäten analytischer Funktionen, Monatsh. Math. Phys. 29 (1918), pp. 3-47.

† Examples of single-valued and analytic functions whose singular points form a perfect discontinuous set but which remain less than a constant have been constructed by Denjoy and Pompéju.

STOCKHOLM,
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